Linear Solvability in the Viewing Graph

Alessandro Rudi, Matia Pizzoli, Fiora Pirri

Department of Computer and System Sciences, Sapienza University of Rome

Abstract. The Viewing Graph [1] represents several views linked by the corresponding fundamental matrices, estimated pairwise. Given a Viewing Graph, the tuples of consistent camera matrices form a family that we call the Solution Set.
This paper provides a theoretical framework that formalizes different properties of the topology, linear solvability and number of solutions of multi-camera systems. We systematically characterize the topology of the Viewing Graph in terms of its solution set by means of the associated algebraic bilinear system. Based on this characterization, we provide conditions about the linearity and the number of solutions and define an inductively constructible set of topologies which admit a unique linear solution. Camera matrices can thus be retrieved efficiently and large viewing graphs can be handled in a recursive fashion. The results apply to problems such as the projective reconstruction from multiple views or the calibration of camera networks.

1 Introduction

In this paper we extend the notion of solvability for a Viewing Graph as given in [1], namely the graph relating several views accounted for by fundamental matrices. We introduce the notion of Solution Set together with a new taxonomy taking into account both linear solvability and the number of solutions of the Viewing Graph. In particular we introduce an inductively constructible set of topologies admitting a unique linear solution. We also show that the method provided allows for a building blocks design that can be used to inductively construct more complex topologies.

Inductive topologies are very useful to combine global and incremental methods for camera matrix estimation. Our formalisation, beside providing a general theoretical framework, it contributes to the hierarchical and recursive approaches for solving the n-view problem in 3D reconstruction and, building on linear-solving subgraphs, it does not involve choosing between multiple solutions or disambiguating results.

Relying on multiview tensors requires establishing feature correspondences between pairs [2], triples [3] or quadruples [4] of images. The geometry of a n-focal system cannot be described by a tensor for $n > 4$. The fundamental matrix is the most basic tool to estimate camera matrices and it can be used as building block to address complex configurations [5]. It can be conveniently estimated from pairs of views and constitutes a less redundant representation than higher order tensors. On the other hand, fundamental matrices for triples of views should be compatible [6], a condition that is satisfied in the case of the trifocal tensor.

Automatic approaches to the computation of structure and motion involve an estimation of camera matrices to initialise the bundle adjustment [7]. The problem of how to initialise the estimation, the order and techniques by which views can be added to the existing ones has been widely faced in literature and several solutions have been proposed. In particular, global methods, based on factorisation [8, 9], allow for the computation of sets of views. These methods require to compute feature correspondences across the entire sequence. On the other hand, hierarchical approaches [10] overcome this limitation but introduce the need to merge subsequences in consistent projective frames.

The *Viewing Graph* was introduced by Levi and Werman in [1] to model the bifocal constraints from pairs of views. They studied different topologies and obtained methods to linearly compute the fundamental matrices that can complete the graph. In [11] camera triplets from fundamental matrices constitute the basic subgraphs used to inductively solve triangular topologies using the linearity results from the Viewing Graph.

In this work we start from very well known results on the computation of projective camera matrices and provide a theoretical framework with two aims: to collect and systematise the above results about topologies and linear solvability in a common formalisation and to characterise the topologies for which the estimation is linear and admits a unique solution. Thus, the proposed method is particularly suitable when dealing with the high changeability and severe occlusions that characterise for example the unstructured image datasets collected by web crawling. The paper is organised as follows. We first introduce the *Viewing Graph* and define the *Extended* and *Non Redundant Solution Set*, giving also a sufficient condition for their non-emptiness. In Section 3 a characterisation of the two solution sets is derived in terms of the *Viewing Graph System*; different conditions on the number of solutions and on the linear solvability are formulated and the *Linear Minimal Solution Superset* and the *Linear Maximal Viewing Subgraph* are introduced. Both the taxonomy of the Viewing Graphs and the inductive construction of *Linearly Solvable Viewing Graphs* are addressed in Section 4. Finally, in Section 5, the application to projective reconstruction is sketched and the conclusions are drawn.

2 The Viewing Graph

According to [1] an N-view scene can be represented as a graph $\mathcal{G} = (V, E)$ whose nodes V are the views and whose edges E are the fundamental matrices between the views. Levi and Werman in [1] are concerned with the following problems: 1. Given a subset $E' \subset E$, what further edges can be computed using only E'? 2. Which are the graphs $\mathcal{G}$ such that, given $\mathcal{G}$ and $E' \subset E$, E can be identified univocally? They give algorithms to solve graphs up to 6 views.

2.1 The Solution Set of the Viewing Graph

We introduce here a weaker notion of solving graph [1], namely, the notion of *Solution Set* of a Viewing Graph.

Definition 1 (Solution Set). *Given a viewing graph $\mathcal{G} = (V, E)$, a solution set is the set of n-tuples of camera matrices $\langle P_1, \ldots, P_n \rangle$ which satisfy the constraints associated with the fundamental matrices F_{ij} in E.*

Let $\wp_{\mathcal{G}}$ be the set of all n-tuples of camera matrices which solve the Viewing Graph $\mathcal{G}$. We have that, for any projective transformation Z and for any n-tuple $t \in \wp_{\mathcal{G}}$, the tuple tZ, obtained applying Z to any camera matrix in t, is a solution for the system and so is $tZ \in \wp_{\mathcal{G}}$. Moreover, since any projective matrix is defined modulo a scale factor, if we have that $t = (P_1, ..., P_n) \in \wp_{\mathcal{G}}$ then we have that $t' = (\lambda_1 P_1, ..., \lambda_n P_n) \in \wp_{\mathcal{G}}$ for any $\lambda_i \neq 0$ as well.

In order to avoid these redundancies of representation we are interested in finding only the set $\Psi_{\mathcal{G}} = \left(\wp_{\mathcal{G}}/GL(4)\right)/\mathbb{R}^{\star}$ of all the orbits of $\wp_{\mathcal{G}}$ under the action of the group of projective transformations and element-wise multiplication by a scalar.

Definition 2 (Extended and Non-Redundant Solution Set). *Given a Viewing Graph $\mathcal{G}$, the* extended Solution Set *is the set of all n-tuples of camera matrices which satisfy the constraints imposed by the fundamental matrices in E. We denote this set by $\wp_{\mathcal{G}}$. The quotient set of $\wp_{\mathcal{G}}$, with respect to projective transformation and scalar multiplication, is the* non-redundant Solution Set*, which we denote by $\Psi_{\mathcal{G}}$.*

Note that, for practical purposes, only $\Psi_{\mathcal{G}}$ is of interest.

We shall state now a sufficient condition for the existence of a solution set. We recall that three fundamental matrices F_{ij}, F_{kj} and F_{ki} are said to be *compatible* if they satisfy the following conditions:

$$\mathbf{e}_{ik}^{\top} F_{ij} \mathbf{e}_{jk} = \mathbf{e}_{kj}^{\top} F_{ki} \mathbf{e}_{ij} = \mathbf{e}_{ki}^{\top} F_{kj} \mathbf{e}_{ji} = 0 \tag{1}$$

with $\mathbf{e}_{ij} \neq \mathbf{e}_{ik}$ the non collinearity conditions for the camera centers [6]. Here $\mathbf{e}_{ij}$ is the epipole arising in view i from view j.

Theorem 1 (Existence of a solution). *Let $\mathcal{G} = (V, E)$ be a Viewing Graph on n views. If all the triples of fundamental matrices satisfy the compatibility condition (1) then there exists a non empty solution set for $\mathcal{G}$.*

Proof. Let $\mathcal{F}_{ijk}$ be the set $\{F_{ij}, F_{ik}, F_{jk}\}$ related to the views v_i v_j and v_k. Note that, from any initial set of compatible triples we can build a solution for $\mathcal{G}$ recursively starting from a triple of fundamental matrices, finding its solution and then adding to the solution an unsolved view at a time. The construction of a solution is illustrated in Section 4.1.

□

In the following a Viewing Graph G, in which all triples of fundamental matrices satisfy the compatibility condition, is said to be a *fully compatible* Viewing Graph.

3 The Viewing Graph System

In this section we give a characterisation of the *Extended* and *Non-Redundant* solution sets of a viewing graph and, in particular, we show both the conditions to obtain a linear solution and a simple computation method.

Let us consider Triggs's F-e closure [12]:

$$\textbf{Two views equation:} \qquad F_{12}\, P_1 + \gamma_{12}\, [\mathbf{e}'_{12}]_\times\, P_2 = 0 \tag{2}$$

where F_{12} is the fundamental matrix relating the camera matrices P_1 and P_2, $\mathbf{e}'_{12}$ is the left epipole, $[\cdot]_\times$ is the cross matrix operator and γ_{12} is a free scale parameter. Since the scale parameter is made explicit, the equation carries 8 constraints and 23 degrees of freedom due to the two camera matrices P_1, P_2, including their scale and the overall scale parameter γ_{12}. The *two views equation* (2) completely defines two camera matrices given a fundamental matrix, modulo a projective transformation.

3.1 Characterisation of the extended and non-redundant solution sets

Let $\mathcal{G} = (V, E)$ be a Viewing Graph and, for homogeneity of representation, let us define $A_{ij} = F_{ij}$ and $B_{ij} = \left[\mathbf{e}'_{ij}\right]_\times$, for any $F_{ij} \in E$. The extended solution set $\wp_\mathcal{G}$ is characterised by the set of two views equations for any F_{ij} related to the constraints it imposes, namely:

$$\wp_\mathcal{G} = \left\{(P_1, \ldots, P_n) \mid A_{ij}P_i + \gamma_{ij}B_{ij}P_j = 0, \forall ij \text{ such that there exists } F_{ij} \in E\right\} \tag{3}$$

On the other hand, the quotient set $\Psi_\mathcal{G}$ can be characterised choosing an arbitrary projective frame to which all the solutions should belong. In particular, let $F_{12} \in E$, we select the projective frame in which $P_1 = [I|0]$ and $P_2 = \left[[\mathbf{e}'_{12}]_\times F_{12}|\mathbf{e}'_{12}\right]$.

Moreover, since any projective matrix is defined modulo a scale factor, for any P_i one of the γ_{ij} is redundant, hence for any i one of γ_{ij} can be set to 1. The non-redundant system characterising the set $\Psi_\mathcal{G}$ is:

$$\begin{cases} A_{ij}P_i + \gamma_{ij}B_{ij}P_j = 0, \forall ij \text{ such that there exists } F_{ij} \in E, \text{ with } F_{ij} \neq F_{12}\} \\ P_2 = \left[[\mathbf{e}'_{12}]_\times F_{12}|\mathbf{e}'_{12}\right] \\ P_1 = [I|0] \\ \gamma_{i\,k(i)} = 1 \qquad\qquad \forall i \in \{3..n\} \end{cases} \tag{4}$$

here $k(i)$ is a total function defined on $\{3..n\}$.

As we can see, the characterisation of the two Solution sets (3)(4) are algebraic bilinear systems for which, in general, there is no known solution except for very simple special cases [13]

Definition 3 (Linear and Non-Linear Viewing Graph System). *Given a non-redundant Solution Set $\Psi_{\mathcal{G}}$, the system associated with $\Psi_{\mathcal{G}}$ is the* Viewing Graph System*, $\Sigma_{\mathcal{G}}$. Whenever $\Sigma_{\mathcal{G}}$ is linearly solvable then $\Sigma_{\mathcal{G}}$ and $\mathcal{G}$ are, respectively, the* Linear Viewing Graph System *and the* Linear Viewing Graph*. Given a Two Views Equation $E_{uv} \in \Sigma_{\mathcal{G}}$, of a Linear Viewing Graph System, this must be equivalent to one of the following three forms:*

$$\begin{array}{ll} \Omega_{ij} : A_{ij}P_i + B_{ij}P_j & = 0 \\ \Delta_{\kappa l} : A_{\kappa l}P_\kappa + B_{\kappa l}P_l & = 0 \\ \Lambda_{i\kappa} : A_{i\kappa}P_i + \gamma_{i\kappa}B_{i\kappa}P_\kappa & = 0 \end{array} \tag{5}$$

Here $u, v \in \{1, \ldots, n\}, k = \{1, 2\}, i, j \in \{3, \ldots, n\}$ and P_k the two constant projective matrices.

When $\Sigma_{\mathcal{G}}$ and $\mathcal{G}$ are non-linearly solvable they are, respectively, the Non-linear Viewing System *and the* Non-Linear Viewing Graph*.*

An example of this kind of systems (and of the choice of $k(i)$) is illustrated in Section 4 for the Base Case I, II and III.

At this point we are ready to discuss the condition for a Viewing Graph System to be linear, the number of solutions and the induced properties on the Viewing Graphs.

Theorem 2 (Unique solution for the Viewing Graph Topology). *Let $\mathcal{G}$ be a Viewing Graph of m edges and $n+2$ views. Let $\Sigma_{\mathcal{G}}$ be the related non-redundant Viewing Graph System.*
If $\Sigma_{\mathcal{G}}$ has a unique solution then $m \geq \left\lceil \frac{11}{7}n - \frac{15}{7} \right\rceil$.

Proof. $\Sigma_{\mathcal{G}}$ has, by definition, m Two Views Equations in n unknown projective matrices and $m - n$ unknown scale parameters as in (4). Let χ_Σ and δ_Σ be, respectively, the number of constraints and degrees of freedom of $\Sigma_{\mathcal{G}}$. We note that $\chi_\Sigma = 8m$ because any Two View Equation carries 8 constraints and $\delta_\Sigma = 12n + 1(m - n)$, due to the unknown projective matrices and scale factors. Hence $7m \geq 11n$. Note that the inequality states that there should be enough fundamental matrices in order to constrain the degrees of freedom of the unknown camera matrices. Therefore if $\Sigma_{\mathcal{G}}$ has a unique solution it must be also that $\mathcal{G}$ has $m \geq \left\lceil \frac{11}{7}n - \frac{15}{7} \right\rceil$ edges.

□

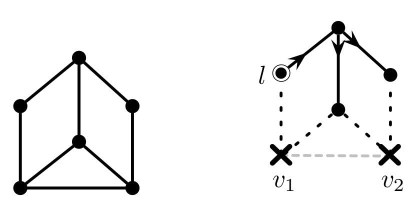

Fig. 1. The Graph $\mathcal{G}$, on the left, satisfies the condition of linear solvability. On the right the construction required by the condition.

We show, now, the conditions for a viewing graph to be linearly solvable. Consider Figure 1, this illustrates a Viewing Graph that can undergo a construction inducing linear solvability. Linear solvability, indeed, requires a suitable assignment of the scale factor, therefore the Viewing Graph that can be adjusted so as to guarantee this assignment is linearly solvable. More specifically, consider the two constant views v_1 and v_2 associated with the constant cameras, which therefore must be connected by an edge, and their neighbours. Let us define the *open neighbour set* $n(v_1, v_2)$ to be the set of all nodes connected to v_1 and v_2, excluding v_1 and v_2. Let us cut all the edges between any node $v_h \in n(v_1, v_2)$ and v_1, v_2. Take any node $l \in n(v_1, v_2)$. If it is possible, starting from l to build a unique path from l to all the nodes in $V - \{v_1, v_2\}$ then we can use this path to ensure that the scale factor is inherited, through l, to all the nodes. If this construction is possible, this means that the graph $\mathcal{H}(v_1, v_2)$, illustrated on the right of Figure 1 can be obtained, hence the starting graph $\mathcal{G}$, as illustrated on the left of the figure, is a linear Viewing Graph.

More formally:

Theorem 3. *Let $\mathcal{G} = (V, E)$ be a Viewing Graph with $n \geq 3$ views. Let $n(v_1, v_2)$ be the union of the open neighbourhood of $v_1, v_2 \in V$, let $\mathcal{H}(v_1, v_2) = \mathcal{G}[V - \{v_1, v_2\}]$ the induced subgraph of $\mathcal{G}$ over the vertex set $V - \{v_1, v_2\}$.*

If there exist $v_1, v_2 \in V$, $l \in n(v_1, v_2)$ and a unique orientation for each edge in $\mathcal{H}(v_1, v_2)$ such that:

- *the in-degree of any node in $V - \{v_1, v_2, l\}$ is 1 and the in-degree of l is 0,*

then $\mathcal{G}$ is a Linearly Solvable Viewing Graph.

Proof. We show the statement by constructing directly the linear system according to Definition 3 Now we prove by construction that if exist $v_1, v_2, l \in V$ and $\mathcal{H}(v_1, v_2)$ that satisfy the hypothesis, then the Viewing Graph System associated to $\mathcal{G}$ is linear.

Suppose that $v_1, v_2, l \in V$ and $\mathcal{H}(v_1, v_2)$ exist and satisfy the statement. First of all, we note that $\mathcal{H}(v_1, v_2)$ has at most $n - 3$ edges, because any node except l can have at most an entering edge and the nodes in $\mathcal{H}(v_1, v_2)$ are $n - 2$. Then we proceed in the construction of a Viewing Graph System Σ. We start from a system Σ which contains only the equations $P_2 = \left[[e'_{12}]_\times F_{12} | e'_{12}\right]$ and $P_1 = [I|0]$ For any edge e_{ij} (edge from v_i to v_j) in $\mathcal{H}(v_1, v_2)$ we add in Σ the equation Ω_{ij}. We choose an edge from $\mathcal{G}$ which connects l to v_κ, with $\kappa \in \{1, 2\}$ and add to Σ the equation $\Delta_{\kappa l}$. For any other edge $e_{i\kappa}$ which connects the nodes v_1, v_2 to their neighbourhood $n(v_1, v_2)$ we add to Σ the equation $\Lambda_{i\kappa}$. The equations Ω, Δ, Λ are, thus, as in Definition 3. The system, specified by Ω, Δ, Λ, mentions the two equations defining P_1 and P_2. For any edge in $\mathcal{G}$ there is the related equation in $\Sigma_\mathcal{G}$ and for any camera matrix P_i, with $i = 3 \ldots n$, the scale parameter is set to 1 (the missing parameter γ_{ij} in the equation Ω_{ij} and $\gamma_{\kappa l}$ in the equation $\Delta_{\kappa l}$).

Thus, according to Definition 3, $\Sigma_\mathcal{G}$ is a linear Viewing Graph System.

□

We can note that this sufficient condition allows us to speak directly about linear solvability of a Viewing Graph $\mathcal{G}$ from a topological point of view without using the related algebraic representation $\Sigma_{\mathcal{G}}$

Resolution of a general linear Viewing Graph System Let Σ be a non-redundant Viewing Graph System composed by m Two Views Equations in n unknown camera matrices $P_3, \ldots, P_{n+3}$ and $r = n - m$ scale parameters represented by the vector $\Gamma = \left(\gamma_{g(1)}, \ldots, \gamma_{g(r)}\right)$, where g is a function which associates the position in the vector Γ to the indices of the fundamental matrix related to that scale parameter and g^{-1} its inverse. The solutions can be found vectorising the system such that it assumes the form $Ax = b$ where A and b are suitable constant matrices and x is the unknown vector. In that case the space of all solutions is $x = A^{+}b + null(A)z$, with z a free vector of suitable dimension.

In order to transform Σ in the form $Ax = b$ first of all we represent the unknowns of Σ in the form of a vector $x = \left(vec(P_3)^{\top}, \ldots, vec(P_{n+3})^{\top}, \gamma_{g(1)}, \ldots, \gamma_{g(r)}\right)^{\top}$ of length $12n + r$ and define two matrices $selP_i = (\varphi_{n,i} \otimes I_{12\times 12}, \mathbf{0}_r)$, $sel\gamma_{ij} = \left(\mathbf{0}_{12n}, \varphi_{r,g^{-1}(ij)}\right)$ useful to select respectively only the matrix P_i from x and only the scale parameter γ_{ij}. Here vec is the vectorisation operator, $\otimes$ is the Kronecker product, $\varphi_{n,i}$ is a row vector of length n with all components equal to 0 excepts the i-th which is 1, $\mathbf{0}_n$ is the row vectors of length n of all zeros and $I_{n\times n}$ the identity matrix of dimension $n \times n$.

Thus, to transform Σ in the form $Ax = b$, we simply rewrite the equation in (5) of Definition 3, with respect to x as follows:

$$\begin{array}{ll} \Omega_{ij} \equiv \{(I_{4\times 4} \otimes A_{ij})\, selP_i \;+\; (I_{4\times 4} \otimes B_{ij})\, selP_j\}\, x = 0 \\ \Delta_{\kappa i} \equiv \{(I_{4\times 4} \otimes B_{\kappa i})\, selP_i\}\;\, x = vec\,(A_{\kappa i} P_{\kappa}) \\ \Lambda_{i\kappa} \equiv \{(I_{4\times 4} \otimes A_{i\kappa})\, selP_i \;+\; vec\,(B_{i\kappa} P_{\kappa})\, sel\gamma_{i\kappa}\}\, x = 0 \end{array} \tag{6}$$

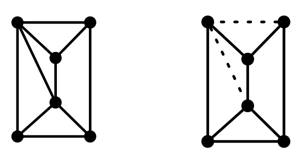

Fig. 2. Given a nonlinear Viewing Graph $\mathcal{G}$ on the left, the associated *Linear Maximal Viewing Subgraph* $\Gamma^{\star}_{\mathcal{G}}$ on the right.

3.2 The Linear Minimal Solution Superset of a nonlinear Viewing Graph

Given a Nonlinear Viewing Graph $\mathcal{G}$ we are interested in simplifying as much as possible the search for the solution set $\Psi_{\mathcal{G}}$. In order to do so we introduce the Linear Minimal Superset of $\Psi_{\mathcal{G}}$ and the Linear Maximal Viewing Subgraph of $\mathcal{G}$.

Given a Viewing Graph $\mathcal{G} = (V, E)$ and the associated Viewing Graph System $\Sigma_{\mathcal{G}}$, an edge e in $\mathcal{G}$ corresponds to a constraint c in $\Sigma_{\mathcal{G}}$. This means that removing

e from $\mathcal{G}$, and so obtaining $\mathcal{G}' = (V, E - \{e\})$, then the corresponding Viewing Graph System $\Sigma_{\mathcal{G}'}$ is less constrained than $\Sigma_{\mathcal{G}}$ indeed $\Sigma_{\mathcal{G}'} = \Sigma_{\mathcal{G}} - \{c\}$. Thus $\Psi_{\mathcal{G}} \subseteq \Psi_{\mathcal{G}'}$.

Definition 4 (Linear Maximal Viewing Subgraph). *Given a Viewing Graph $\mathcal{G}$, and letting $lin(\mathcal{G})$ be the set of all the Linear Viewing Subgraphs of $\mathcal{G}$, the Linear Maximal Viewing Subgraph of $\mathcal{G}$ is the Viewing Graph $\Gamma^\star_{\mathcal{G}} \in lin(\mathcal{G})$, such that the associated linear solution set is minimal :*

$$\Gamma^\star_{\mathcal{G}} = \arg \min_{\Gamma \in lin(\mathcal{G})} |\Psi_\Gamma|. \tag{7}$$

See Fig. 2. The definition is well-posed, indeed given a Nonlinear Viewing Graph $\mathcal{G} = (V, E)$ we can always remove edges from $\mathcal{G}$ in order to obtain a linearly solvable Viewing Subgraph $\mathcal{L} = (V, E')$ with $E' \subset E$ and so $\Psi_{\mathcal{G}} \subseteq \Psi_{\mathcal{L}}$. This means that a Linear Viewing Subgraph of a Viewing Graph System $\mathcal{G}$ always exists (for example $\mathcal{L} = (V, \emptyset)$).

Definition 5 (Linear Minimal Solution Superset). *Given a Viewing Graph $\mathcal{G}$, the Linear Minimal Solution Superset of $\mathcal{G}$, $\Pi_{\mathcal{G}}$, is the Solution Set of the Linear Maximal Viewing Subgraph $\Gamma^\star_{\mathcal{G}}$*

$$\Pi_{\mathcal{G}} = \Psi_{\Gamma^\star_{\mathcal{G}}} \tag{8}$$

The Linear Minimal Solution Superset of a nonlinear Viewing Graph is important because we have always that $\Psi_{\mathcal{G}} \subseteq \Pi_{\mathcal{G}}$ and $\Pi_{\mathcal{G}}$ is linearly computable. As a consequence, when we are searching for a solution to $\mathcal{G}$ we have only to search in the minimal linear space $\Pi_{\mathcal{G}}$ instead of in the huge Cartesian product of all camera matrices.

Moreover, when $\Pi_{\mathcal{G}}$ contains only a solution and the fundamental matrices expressed by $\mathcal{G}$ are fully compatible, we have that $|\Pi_{\mathcal{G}}| = 1$, $|\Psi_{\mathcal{G}}| \geq 1$, $\Psi_{\mathcal{G}} \subseteq \Pi_{\mathcal{G}}$ and thus $\Psi_{\mathcal{G}} = \Pi_{\mathcal{G}}$. This means that in this case we have only to solve the linear graph $\Gamma^\star_{\mathcal{G}}$ to have the solution of the bigger and possibly nonlinear $\mathcal{G}$.

4 Topology and Solvability

First of all we collect from the previous sections some results about the Topology of a Viewing Graph, the linearity and the number of solutions of the related Viewing Graph System.

Any Viewing Graph with m edges and n nodes where $m < \left\lceil \frac{11}{7}n - \frac{15}{7} \right\rceil$ and any linear Viewing Graph whose system is underdetermined has a family of solutions in the Solution Set.

Any full compatible linear or nonlinear Viewing Graph $\mathcal{G}$ for which $|\Pi_{\mathcal{G}}| = 1$ admits only one linear solution.

Finally we can introduce the taxonomy of the set of Viewing Graphs in terms of number of solutions and linear solvability (fig. 3).

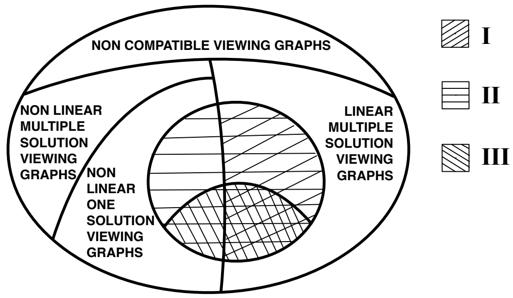

Fig. 3. Taxonomy of the Set of Viewing Graphs. I) One Solution Linear Viewing Graphs; II) Viewing Graphs whose Linear Maximal Subgraph has one solution; III) Viewing Graphs whose Linear Maximal Subgraph belongs to the set of the Recursive Topologies we presented in subsection 4.1.

4.1 An Inductively Constructible Set of One Solution Linear Viewing Graphs

If we are able to solve a linear Viewing Graph Γ which admits a unique solution then we are able to solve all Viewing Graphs for which Γ is the Linear Maximal Viewing Subgraph.

Thus, it is important to find a cheap way to span the space of topologies that lead to Linear Viewing Graphs with unique solution. With this aim, in the following we give a characterization of a constructible subset of the set of the *One Solution Linear Viewing Graphs.*

We apply the results from the previous sections.

Base Linear Case I: Three Views Consider the Viewing Graph $\mathcal{G} = (V, E)$ given by three views $V = (v_1, v_2, v_3)$ and the three fundamental matrices linking them $E = (F_{12}, F_{23}, F_{31})$ (fig. 4). This topology satisfies the necessary and sufficient conditions in section 3 and so it is linear. In addition, when the camera matrices are in general position, it admits at most one solution. When it satisfies the full compatibility condition then $\mathcal{G}$ has a unique solution. This is a well known result [6, 1]. The interesting thing here is that we demonstrated it only using the tools from the previous sections.

For completeness, we algebraically find a close solution for $\mathcal{G}$. The orbit set of solutions $\Psi_{\mathcal{G}}$ is represented by the related non-redundant Viewing Graph System as follows:

$$\begin{cases} A_{23}P_2 + B_{23}P_3 = 0 \\ A_{31}P_3 + \gamma_{31}B_{13}P_1 = 0 \\ P_1 = [I|0] \\ P_2 = \left[[\mathbf{e}'_{12}]_\times F_{12}|\mathbf{e}'_{12}\right] \end{cases} \tag{9}$$

where in the first equation γ_{23} has been set to 1 in order to disambiguate the scale of P_3 with respect of P_2 (see proof of the sufficiency condition in Section 2.1). Thus the system is linear; in general it is overdetermined but, when the fundamental matrices are compatible, it admits at least one solution which can be stated in closed form:

$$\begin{cases} P_1 = [I|0] \\ P_2 = \left[[e'_{12}]_\times F_{12}|e'_{12}\right] \\ P_3 = \begin{pmatrix} B_{23} \\ A_{31} \end{pmatrix}^+ \begin{pmatrix} A_{23}P_2 \\ \gamma_{23}B_{31}P_1 \end{pmatrix} \end{cases} \tag{10}$$

where $(\cdot)^+$ is the pseudo-inverse operator and $\gamma_{23} = -\frac{cd^\top}{dd^\top}$, with $c = UA_{23}P_2$, $d = VB_{31}P_1$ and $(U\,V) = null\begin{pmatrix} B_{23} \\ A_{31} \end{pmatrix}$.

Base Linear Case II: Five Views Let $\mathcal{G} = (V, E)$ be the Viewing Graph with $V = (v_1, \ldots, v_5)$ and $E = (F_{12}, F_{31}, F_{41}, F_{25}, F_{35}, F_{45})$ as in figure (fig. 4). This topology satisfy the conditions of section 3 so that it's linear and, when the camera matrices are in general position, it admits at most one solution. When it satisfies the full compatibility condition, then $\mathcal{G}$ has a unique solution [1].

For completeness: the set of solutions $\Psi_\mathcal{G}$ is represented by the following Linear Viewing Graph System

$$\begin{cases} A_{31}P_3 + \gamma_{31}B_{31}P_1 = 0 \\ A_{41}P_4 + \gamma_{41}B_{41}P_1 = 0 \\ A_{25}P_2 + B_{25}P_5 = 0 \\ A_{54}P_5 + B_{54}P_4 = 0 \\ A_{53}P_5 + B_{53}P_3 = 0 \\ P_1 = [I|0] \\ P_2 = \left[[e'_{12}]_\times F_{12}|e'_{12}\right] \end{cases} \tag{11}$$

here $\gamma_{25}, \gamma_{54}, \gamma_{53}$ have been set to 1 in order to disambiguate the scale respectively of P_5 with respect to P_2, P_4 with respect to P_5 and P_3 with respect to P_5 (see the proof in Section 2.1).

Base Linear Case III: Six Views Let $\mathcal{G} = (V, E)$ be the Viewing Graph with $V = (v_1, \ldots, v_6)$ and $E = (F_{32}, F_{52}, F_{61}, F_{12}, F_{13}, F_{34}, F_{45}, F_{56})$ as in figure (fig. 4).

Even in this case $\mathcal{G}$ satisfies the two topological conditions of section 2.1. Consequently, it is linearly solvable. If the camera matrices are in general positions it admits at most a solution which exist when the full compatibility holds (see [1])

For completeness: the set of solutions $\Psi_\mathcal{G}$ is represented by the following Linear Viewing Graph System

$$\begin{cases} A_{32}P_3 + \gamma_{32}B_{32}P_2 = 0 \\ A_{52}P_5 + \gamma_{52}B_{52}P_2 = 0 \\ A_{61}P_6 + \gamma_{61}B_{61}P_1 = 0 \\ A_{13}P_1 + B_{13}P_3 = 0 \\ A_{34}P_3 + B_{34}P_4 = 0 \\ A_{45}P_4 + B_{45}P_5 = 0 \\ A_{56}P_5 + B_{56}P_6 = 0 \\ P_1 = [I|0] \\ P_2 = \left[[\mathbf{e}'_{12}]_\times F_{12}|\mathbf{e}'_{12}\right] \end{cases} \tag{12}$$

here $\gamma_{13}, \gamma_{34}, \gamma_{45}, \gamma_{56}$ have been set to 1 in order to disambiguate the scale respectively of P_3 with respect to P_1, P_4 with respect to P_3, P_5 with respect to P_4 and P_6 with respect to P_5 (see the proof in Section 2.1).

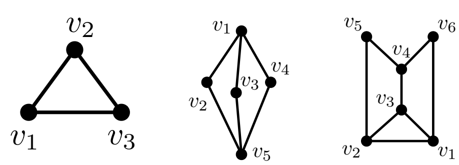

Fig. 4. In order: Base Linear Case I, Case II, Case III.

Composition rule	**I**	**II**	**III**	**IV**
	Merging two solved graphs which share two views	Merging two solved graphs which share a view and are linked by an edge	Adding a new double connected view	Merging two solved graphs with three edges

Composition Rule I: Merging two solved graphs which share two views
Let $\mathcal{G}$ be a Viewing Graph composed by two already solved graphs Γ and Υ which share two views v_1 and v_2. The solution of $\mathcal{G}$ $t_{\mathcal{G}}$ can be built from the solutions $t_\Gamma = (P_1, P_2, \dots)$ and $t_\Upsilon = (P'_1, P'_2, \dots)$ (Fig. 5). Indeed, since the cameras P_1, P_2 and P'_1, P'_2 are linked by the same fundamental matrix F_{12}, a projective transformation exists that maps P_i on P'_i for $i = 1, 2$. We can find it simply by

$$Z = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}^+ \begin{pmatrix} P'_1 \\ P'_2 \end{pmatrix}$$

Once we have Z we have to right multiply all the camera matrices in t_Γ in order to express them in the same projective frame of Υ such that P_i and P'_i coincide for $i = 1, 2$.

Composition Rule II: Merging two solved graphs which share a view and are linked by an edge Let $\mathcal{G}$ be a Viewing Graph composed by two already solved graphs $\Gamma = (V, E)$ and $\Upsilon = (W, L)$ which share a view v_1 and are linked by a fundamental matrix F_{23} with the related views $v_2 \in V$ and $v_3 \in W$ (fig. 5). We can solve $\mathcal{G}$ by finding the solution to $\Theta = (\{v_1, v_2, v_3\}, \{F_{12}, F_{13}, F_{23}\})$ through Base Case I and then apply Composition Rule I two times, first to Γ, Θ and then to $(\Gamma \cup \Theta), \Upsilon$.

Composition Rule III: Adding a new double connected view Let $\mathcal{G}$ be a Viewing Graph with n views composed by an already solved graph $\Gamma = (V, E)$ of $n-1$ views connected by F_{1n}, F_{2n} to a view v_n and let $t = (P_1, \dots, P_{n-1})$ the solution for Γ (Fig. 5). We are able to solve the Viewing Graph $\mathcal{G}$ as follows. First of all we calculate F_{12} by P_1 and P_2 from t. Then we solve the Viewing Graph System related to the Viewing Graph $\Upsilon = (\{v_1, v_2, v_n\}, \{F_{12}, F_{1n}, F_{2n}\})$ using the Base Case I and then we merge the two already solved Viewing Graphs Γ, Υ using the Compositional Rule I.

Composition Rule IV: Merging two solved graphs with three edges Let $\mathcal{G}$ be a Viewing Graph composed by two already solved graphs $\Gamma = (V, E)$ and $\Upsilon = (W, L)$ which are linked by three fundamental matrices F_{14}, F_{25}, F_{36} with the related views $v_1, v_2, v_3 \in V$ and $v_4, v_5, v_6 \in W$ (Fig. 5). Supposing that Γ and Υ are already solved, we can calculate F_{12}, F_{23}, F_{13} from Γ and F_{45}, F_{56} from Υ. Then we can solve $\mathcal{G}$ by finding the solution to $\Theta = (H, K)$ with $H = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $K = \{F_{12}, F_{23}, F_{13}, F_{45}, F_{56}, F_{14}, F_{25}, F_{36}\}$ through Base Case III and then apply Composition Rule I two times, first to Γ, Θ and then to $(\Gamma \cup \Theta), \Upsilon$.

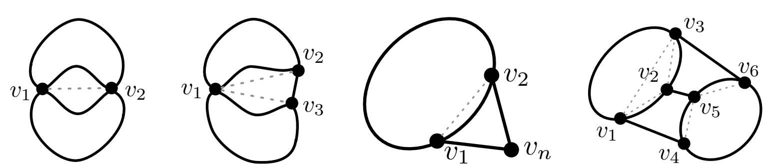

Fig. 5. In order: Compositional Rule I, Rule II, Rule III, Rule IV.

In this subsection we substantially sketched a compositional topology that allows to compute the solution to the Linear Maximal Viewing Subgraph of a given Graph $\mathcal{G}$ in a bottom-up fashion. We can solve arbitrarily chosen parts of this subgraph and merge them in arbitrary order (in agreement with the conditions of the composition rules) arriving to the same result, because of the linearity of the problem under the sufficiency condition of Theorem 1, at least in the unique solution case.

In the end, due to this invariance to the order, this incremental bottom-up linear approach makes the problem of choosing a right view order less critical.

5 Conclusions and discussion

This paper integrates in a common framework previous results on the estimation of camera matrices from unstructured collections of views with a characterization of a subclass of topologies for which the solution is linear and unique. The Viewing Graph has been equipped with its algebraic counterpart, the Viewing Graph System. This characterization provides a sufficient condition for the linear solvability of the system of bilinear equations and the associated Viewing Graph. Indeed, we translated the linear solvability condition to be directly applied to the topology of Viewing Graphs, bridging from the algebraic to the graph based representation. In future works we are going to use the tools described in this paper to develop heuristic and approximated graph algorithms operating on Viewing Graphs in order to compute the Linear Maximal viewing Subgraph and find the Minimal Solution Set. A similar approach is particularly suitable to deal with the high changeability and severe occlusions that characterize, for example, the large, unstructured image datasets collected by web crawling.

Acknowledgement. This research has been funded by EU-FP7 *NIFTI* project.

References

1. Levi, N., Werman, M.: The viewing graph. CVPR (2003)
2. Luong, Q.T., Faugeras, O.: The fundamental matrix: theory, algorithms, and stability analysis. International Journal of Computer Vision **17** (1995) 43–75
3. Torr, P., Zisserman, A.: Robust parameterization and computation of the trifocal tensor. Image and Vision Computing **15** (1997) 591–605
4. Hartley, R.I.: Cheirality. Int. J. Comput. Vision **26** (1998) 41–61
5. Goldberger, J.: Reconstructing camera projection matrices from multiple pairwise overlapping views. Computer Vision and Image Understanding **97** (2005) 283–296
6. Hartley, R.I., Zisserman, A.: Multiple View Geometry in Computer Vision. Second edn. Cambridge University Press, ISBN: 0521540518 (2004)
7. Triggs, B., McLauchlan, P.F., Hartley, R.I., Fitzgibbon, A.W.: Bundle adjustment - a modern synthesis. In: ICCV. (2000)
8. Tomasi, C., Kanade, T.: Shape and motion from image streams under orthography: a factorization method. Int. J. Comput. Vision **9** (1992) 137–154
9. Sturm, P.F., Triggs, B.: A factorization based algorithm for multi-image projective structure and motion. In: ECCV. (1996)
10. Fitzgibbon, A.W., Zisserman, A.: Automatic camera recovery for closed or open image sequences. In: ECCV. (1998)
11. Sinha, S.N., Pollefeys, M., McMillan, L.: Camera network calibration from dynamic silhouettes. In: CVPR. (2004)
12. Triggs, B.: Linear projective reconstruction from matching tensors. Image and Vision Computing **15** (1997) 617–625
13. Cohen, S., Tomasi, C.: Systems of bilinear equations (1998)